A Note on k-Shortest Paths Problem

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Abstract

It is well-known that in a directed graph, if deleting any edge will not affect the shortest distance between two specific vertices s and t, then there are two edge-disjoint paths from s to t and both of them are shortest paths. In this paper, we generalize this to shortest k edge-disjoint s-t paths for any positive integer k.

1 Introduction

Given a directed graph $G = (V, E)$ with weight $w(e)$ on edge $e \in E$, let $s, t \in V$ be two specific vertices. A well-known result is that if after deleting any edge in the shortest path from s to t, there is still an s-t path of the same length, then there are two edge-disjoint paths from s to t and both of them are shortest path. This can be shown, e.g., by Menger's theorem [2] considering the subgraph consisting of all shortest s-t paths.

In this paper, we extend this result to shortest k edge-disjoint paths, given by the following claim.

Theorem 1. *Let $G = (V, E)$ be a directed graph with weight $w(e)$ on each edge $e \in E$ and no cycles of negative weight. Given two specific vertices $s, t \in V$, assume that there are k edge-disjoint paths from s to t. Let $P_1, P_2, \cdots, P_k$ be k edge-disjoint s-t paths so that their length $L \triangleq \sum_{i=1}^{k} w(P_i)$ is minimized, where $w(P_i) = \sum_{e \in P_i} w(e)$. Further, suppose that for every edge $e \in E$, the graph $G - \{e\}$ has k edge-disjoint s-t paths with the same total length L. Then there exist $k+1$ edge-disjoint s-t paths in G such that each of them is a shortest path from s to t.*

Note that the claim implies, in particular, that the original k edge-disjoint s-t paths $P_1, P_2, \cdots, P_k$ are shortest paths. The proof of the theorem involves a careful examination of a specific real-valued min-cost max-flow defined from the arithmetic average of $|E|$ different integer-valued min-cost max-flows and showing that any s-t path with positive amount of flows on each edge forms a shortest path. Details of the proof are given in the next section.

The condition of deleting any edge will not affect the total length of shortest k edge-disjoint paths is motivated from applications in game theory and mechanism design [5]. For example, we can consider all vertices in G by geographical locations and edges by the corresponding paths between them. A shipping company plans to carry k items from one location s to the other t. Due to capacity constraint, every edge can carry at most one item. Further, for each edge, there is an associated cost $c(e)$ (e.g. maintenance) incurred to local people to provide their service. Therefore, the company has to make a payment to each edge it uses to recover those costs. By a standard game theoretical assumption, all edges are selfish and hope to receive as much payment as possible (given that their costs are recovered). In a market setting, each edge set up a price $w(e) \geq c(e)$ asking for the company. Given all $(w(e))_{e\in E}$, naturally the company will purchase k edge-disjoint paths with the smallest total payment, i.e. shortest k edge-disjoint paths with respect to $w(e)$. This defines a game among all edges where the strategy of each edge is the price $w(e)$ it determines. Nash equilibrium [4], where no edge can unilaterally increase its $w(e)$ to receive more payment, captures a stable state of the game and provides a natural solution to the market. In other words, in a Nash equilibrium, if anyone increases its $w(e)$ by any amount, the company will purchase another set of k paths with the same total payment. This is exactly the condition given by the theorem. Our theorem, on the other hand, gives a nice characterization of the marketplace in a Nash equilibrium. Recently, we came to know that Kempe et al. [3] independently showed the same characterization of Nash equilibrium when the graph is composed of $k+1$ edge-disjoint paths.

2 Proof of the Theorem

Given the graph G and integer k, we construct a flow network $\mathcal{N}_k(G)$ as follows: We introduce two extra nodes s_0 and t_0 and two extra edges s_0s and tt_0. The set of vertices of $\mathcal{N}_k(G)$ is $V \cup \{s_0, t_0\}$ and the set of edges is $E \cup \{s_0s, tt_0\}$. The capacity $cap(\cdot)$ and cost per bulk capacity $cost(\cdot)$ for each edge in $\mathcal{N}_k(G)$ is defined as follows:

- $cap(s_0s) = cap(tt_0) = k$ and $cost(s_0s) = cost(tt_0) = 0$.
- $cap(e) = 1$ and $cost(e) = w(e)$, for $e \in E$.

Given the above construction, every path from s to t in G naturally corresponds to a bulk flow from s_0 to t_0 in $\mathcal{N}_k(G)$. Hence, the set of k edge-disjoint paths $P_1, P_2, \ldots, P_k$ in G corresponds to a flow $\mathcal{F}_G$ of size k in $\mathcal{N}_k(G)$. In addition, the minimality of $L = \sum_{i=1}^{k} w(P_i)$ implies that $\mathcal{F}_G$ achieves the minimum cost (which is L) for all *integer*-valued flows of size k, i.e. maximum flow in $\mathcal{N}_k(G)$. Since all capacities of $\mathcal{N}_k(G)$ are integers, we can conclude that $\mathcal{F}_G$ has the minimum cost among all *real* maximum flows in $\mathcal{N}_k(G)$, the details one can find in [1].

For simplicity, we denote the subgraph $G-\{e\}$ by $G-e$. By the fact that for any $e \in E$, the subgraph $G-e$ has k edge-disjoint s-t paths with the same total length L, we know that in the network $\mathcal{N}_k(G-e)$, there still is an integer-valued flow $\mathcal{F}_{G-e}$ of size k and cost L. So $\mathcal{F}_{G-e}$ is also an integer-valued flow of size k and cost L in $\mathcal{N}_k(G)$. Define a real-valued flow in $\mathcal{N}_k(G)$ by $\mathcal{F} = \frac{1}{|E|}\sum_{e\in E}\mathcal{F}_{G-e}$. We have the following observations:

1. It is clear that $\mathcal{F}(e) \leq cap(e)$ for every arc $e \in \mathcal{N}_k(G)$, where $\mathcal{F}(e)$ is the amount of flow on edge e in $\mathcal{F}$, as we have taken the arithmetic average of the flows in the network.
2. $\mathcal{F}$ has cost $\frac{1}{|E|}\sum\limits_{e\in E} cost(\mathcal{F}_{G-e}) = \frac{1}{|E|}\cdot|E|\cdot L = L$.
3. Since $\mathcal{F}_{G-e}(s_0s) = k$ for any $e \in E$, we have $\mathcal{F}(s_0s) = k$. In addition, as each $\mathcal{F}_{G-e}$ is a feasible flow that satisfies all conservation conditions and $\mathcal{F}$ is defined by the arithmetic average of all $\mathcal{F}_{G-e}$'s, we know that $\mathcal{F}$ also satisfies all conservation conditions.

Therefore, $\mathcal{F}$ is a minimum cost maximum flow in $\mathcal{N}_k(G)$. In addition, $\mathcal{F}$ has the following nice property, which plays a fundamental role for the proof:

- For every edge $e \in \mathcal{N}_k(G)$ except s_0s and tt_0, we have $\mathcal{F}(e) \leq cap(e) - \frac{1}{|E|}$, as $\mathcal{F}_{G-e}$ does not flow through e, i.e. $\mathcal{F}_{G-e}(e) = 0$, and $\mathcal{F}_{G-e'}(e)$ is either 0 or 1 for any $e' \in E$.

Let $E_+ = \{e \in \mathcal{N}_k(G) \mid \mathcal{F}(e) > 0\}$. Suppose that there is a path $P' = (e_1, e_2, \ldots, e_r)$ from s_0 to t_0 which goes only along arcs in E_+ and is not a shortest path w.r.t $cost(\cdot)$ from s_0 to t_0 in $\mathcal{N}_k(G)$. Let $\epsilon = \min\left\{\mathcal{F}(e_1), \mathcal{F}(e_2), \ldots, \mathcal{F}(e_r), \frac{1}{|E|}\right\}$. Since $P' \subseteq E_+$, we have $\epsilon > 0$. Let P be a shortest path w.r.t $cost(\cdot)$ from s_0 to t_0 in $\mathcal{N}_k(G)$. Define a new flow $\mathcal{F}'$ from $\mathcal{F}$ by adding ϵ amount of flow on path P and removing ϵ amount of flow from path P'. We have the following observations about $\mathcal{F}'$:

1. $\mathcal{F}'$ satisfies all conservation conditions, as it is a linear combination of three flows from s_0 to t_0: $\mathcal{F}' = \mathcal{F} + \epsilon \cdot P - \epsilon \cdot P'$.

2. The size of flow $\mathcal{F}'$ is k.

3. By the definition of ϵ, the amount of flow of each edge is non-negative in $\mathcal{F}'$. Further, $\mathcal{F}'$ satisfies the capacity constraints. This follows from the facts that $\epsilon \le \frac{1}{|E|}$ and the above property established for $\mathcal{F}$.

4. The cost of $\mathcal{F}'$ is smaller than L because $cost(\mathcal{F}') = cost(\mathcal{F}) - \epsilon(cost(P') - cost(P))$, which is smaller than $L = cost(\mathcal{F})$ as $cost(P) < cost(P')$ by the assumption.

Hence, $\mathcal{F}'$ is a flow of size k in $\mathcal{N}_k(G)$ with cost smaller than $\mathcal{F}$, a contradiction. Thus, every path from s_0 to t_0 in $\mathcal{N}_k(G)$ along the edges of E_+ is a shortest path w.r.t $cost(\cdot)$.

Consider a new network $\mathcal{N}'_{k+1}(G)$ obtained from $\mathcal{N}_{k+1}(G)$ by restricting edges on E_+. (Note that the only difference between $\mathcal{N}_{k+1}(G)$ and $\mathcal{N}_k(G)$ is the capacity on edges s_0s and tt_0, $k+1$ rather than k.) We claim that in this network there is an integer-valued flow of size $k+1$. Suppose otherwise, by max-flow min-cut theorem, there is a cut (S_{s_0}, T_{t_0}) in $\mathcal{N}'_{k+1}(G)$ with size less than or equal to k. By definition, in $\mathcal{N}'_{k+1}(G)$ we have $cap(s_0s) = k+1$ and $cap(tt_0) = k+1$, which implies that $s_0, s \in S_{s_0}$ and $t_0, t \in T_{t_0}$. By the definition of E_+, we know that the total amount of flows in $\mathcal{F}$ on the cut (S_{s_0}, T_{t_0}) is k. Since $\mathcal{F}(e) < 1$ for any edge e of G, we can conclude that there are at least $k+1$ edges from S_{s_0} to T_{t_0} in E_+. This leads to a contradiction, because we have shown that the size of the cut (S_{s_0}, T_{t_0}) is less than or equal to k.

Therefore, we can find an integer-valued flow of size $k+1$ on edges of E_+ in the network $\mathcal{N}_{k+1}(G)$. Such a flow can be thought as the union of $k+1$ edge-disjoint paths from s_0 to t_0. We know that every such path going along edges in E_+ is a shortest path from s_0 to t_0. This in turn concludes the proof, since we have found $k+1$ edge-disjoint shortest paths from s to t in G.

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