On proper colorings of hypergraphs

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1   Introduction

In this paper we consider undirected graphs and hypergraphs. We denote by $V(G)$ the vertex set of a graph G and the edge set by $E(G)$. Notations $v(G)$ and $e(G)$ in our paper stand for the number of vertices and edges respectively.

We denote by $d_G(v)$ the *degree* of vertex $v \in V(G)$ in G. We denote by $\delta(G)$ and $\Delta(G)$ the minimal and maximal vertex degrees of G respectively. We use similar notations ($V(\mathcal{H})$, $E(\mathcal{H})$ and $d_\mathcal{H}(v)$) for a hypergraph $\mathcal{H}$. In this work it is convenient for us to deal with edges and hyperedges in terms of vertex subsets of a graph or a hypergraph.

We denote the *neighborhood* of vertex v in G (i.e., the set of all adjacent to v vertices of G) by $N_G(v)$.

For any set $W \subset V(G)$ we denote by $G(W)$ the *induced subgraph* of G on W (i.e., the subgraph on W that contains all edges of G with two ends in W).

There are several ways to generalize the notation of proper coloring on hypergraphs. For example, strong vertex colorings [1], in which all vertices in every hyperedge have to receive different colors. In the present paper we work with the definition proposed by P. Erdős.

Definition 1. A vertex coloring of a hypergraph $\mathcal{H}$ is called *proper coloring*, if any hyperedge contains at least two vertices of different colors.

In the field of colorings of ordinary graphs many natural questions are still left open. Thus, it is not surprising that vertex colorings of hypergraphs are not well studied. One particular question in the field of hypergraph colorings that has received great attention in the literature (see [4, 5, 6, 7, 8, 10, 11, 12, 13]) is the problem of finding a n-uniform hypergraph with the minimal number of edges $m_k(n)$ that admits no proper vertex k-coloring.

Another problem closely related to the one cited above is the question "what is the minimal n, such that every n-uniform and n-regular hypergraph (i.e., a hypergraph with all edges containing n vertices and all vertices having degree n) admits a proper vertex 2-coloring?" It was shown (see [3]) by means of Lovasz local Lemma and other probabilistic methods, that for $n \geq 9$ every such graph is 2-colorable. Alon and Bregman [2] improved this statement to $n = 8$, and Thomassen [14] has shown finally 2-colorablity for all $n \geq 4$.

The following theorem is the main result of our paper.

Theorem 1. *Let $\mathcal{H}$ be a hypergraph of maximal vertex degree Δ, such that each its hyperedge contains at least δ vertices. Let $k = \lceil \frac{2\Delta}{\delta} \rceil$. Then the following statements hold.*

1) The hypergraph $\mathcal{H}$ admits proper vertex coloring in $k + 1$ colors.

2) The hypergraph $\mathcal{H}$ admits proper vertex coloring in k colors, if $\delta \geq 3$ and $k \geq 3$.

Our theorem gives weaker results than the works cited above, when the minimal size of hyperedge is close to the maximal vertex degree of considered hypergraph. However, for relatively small values of δ with respect to Δ the statement of our theorem becomes interesting. Our proof uses only classic combinatorial methods.

From our main theorem we derive results on dynamic vertex colorings.

Definition 2. A vertex coloring of a graph G is called *dynamic*, if any vertex v of degree at least 2 has at least two vertices of different colors in its neighborhood.

We note that some papers (e.g., [9, 15, 16]) study *proper* dynamic colorings. There it was shown the existence of a proper dynamic vertex coloring of G in $\Delta(G) + 1$ colors [9] and in $\Delta(G)$ colors [16] besides explicitly described series of exceptions. In the current paper we do not require a dynamic coloring to be a proper coloring and obtain the following result.

Theorem 2. *Let G be a graph, $k = \lceil \frac{2\Delta(G)}{\delta(G)} \rceil$. Then the following statements hold.*

1) The graph G admits a dynamic vertex coloring in $k + 1$ colors.

2) The graph G admits a dynamic vertex coloring in k colors, if $\delta(G) \geq 3$ and $k \geq 3$.

2   Hypergraph's Image and Alternating Chains

We further introduce some notions which are important for the following proof of our main result.

Definition 3. We call any graph G (with possible multiple edges) by an *image* of a hypergraph $\mathcal{H}$, if
    (i) $V(G) = V(\mathcal{H})$;
    (ii) there exists a bijection $\varphi : E(G) \rightarrow E(\mathcal{H})$, such that $e \subset \varphi(e)$ for every edge $e \in E(G)$.
    We call φ by the *bijection of image G*.

Remark. *We consider multiple edges of a graph-image G that corresponds to distinct hyperedges of the hypergraph $\mathcal{H}$ as distinct edges.*

As in some classic theorems about vertex colorings, we make use of alternating chains. Next definition will show what we mean by this notion for hypergraphs.
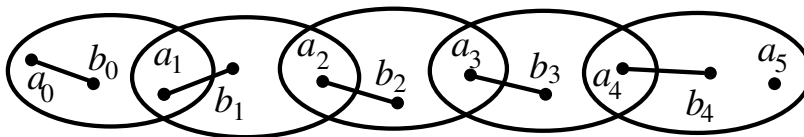

Figure 1: Alternating chain of length 5 with beginning a_0 and end a_5.

Definition 4. Let $\delta \geq 3$ and let G be an image of a hypergraph $\mathcal{H}$. We consider a sequence of vertices $a_0 b_0 a_1 b_1 \ldots a_n$ of $\mathcal{H}$, satisfying the following conditions.

- For each i vertices a_i, b_i, a_{i+1} are different.

- There exist different hyperedges $e_0, \ldots, e_{n-1} \in E(\mathcal{H})$, such that

    - $a_i b_i \in E(G)$ and $\varphi(a_i b_i) = e_i$,
    - $a_i, b_i, a_{i+1} \in e_i$.

Then $a_0 b_0 a_1 b_1 \ldots a_n$ is an *alternating chain* from a_0 to a_n. We say that it has *length n* and that it *goes* through the vertices a_0, b_0, $\ldots$, a_n and through

the edges $a_0b_0,\dots, a_{n-1}b_{n-1}$. We say that this chain *begins* at a_0 and *ends* at a_n.

For two sets $X, Y \subset V(G)$ with $a_0 \in X$ and $a_n \in Y$, we say that $a_0b_0a_1b_1\dots a_n$ is an alternating chain from X to Y.

Remark. 1) *We allow the case $n = 0$ in the definition of alternating chain, that is a_0 is an alternating chain from a_0 to a_0 of length 0.*

2) Since φ is a bijection, then edges $a_1b_1,\dots, a_nb_n$ due to the definition of alternating chain are all different. We recall here that multiple edges corresponding to different hyperedges of G are considered as different edges.

3) Vertices are not necessarily different. An alternating chain may go through some vertices more than once.

Lemma 1. *Let $\mathcal{H}$ be a hypergraph of maximal vertex degree Δ, such that each hyperedge of $\mathcal{H}$ contains at least δ vertices. Let $k = \lceil \frac{2\Delta}{\delta} \rceil$. Then there is an image G of $\mathcal{H}$ with $\Delta(G) \leq k$.*

Proof. Consider a trivial case $\delta = 2$. In this case for any image G of the hypergraph $\mathcal{H}$ it is clear, that $\Delta(G) \leq \Delta = k$. In what follows we assume $\delta \geq 3$.

For a graph H we denote by $V_{k+1}(H)$ the set of all its vertices of degree at least $k + 1$. We denote by $s_{k+1}(H)$ the sum of degrees in the graph H taken over vertices of $V_{k+1}(H)$.

For the sake of contradiction, we assume that the statement of lemma fails. Then for any image H we have $V_{k+1}(H) \neq \varnothing$ and $s_{k+1}(H) > 0$. Let G be an image with the minimal $s_{k+1}(G)$. We denote by φ the bijection of G, and we set $S = V_{k+1}(G)$.

Let U be the set of vertices of G that consists of all possible ends of alternating chains with the beginning in S. We set $F = G(U)$. Clearly, $U \supset S$. In the next we observe some properties of U.

1. *For any edge $e \in E(F)$ the hyperedge $\varphi(e) \subset U$.*
Suppose the contrary. Then $e = uw \in E(F)$ and the hyperedge $\varphi(e)$ contains a vertex $v \notin U$ (see figure 2a). In the following we construct an alternating chain from S to v and, therefore, show that $v \in U$. The latter contradicts our assumption.

We consider the shortest alternating chain $P = a_0b_0 \dots a_n$ from S to $\{u, w\}$. Without loss of generality we may assume that $a_n = u$. Then $a_i \notin \{u, w\}$ for any $0 \leq i < n$. Hence P does not go through $e = uw$. We add to P vertices w, v and obtain an alternating chain from $a_0 \in S$ to v.

2. *If a vertex $u \in U$ is adjacent to a vertex $v \notin U$, then all the vertices of the hyperedge $\varphi(uv)$ except v belong to U.*

Let $u \in U$, $v \notin U$, $uv \in E(G)$, and $e = \varphi(uv)$ be a hyperegde of $\mathcal{H}$. Suppose the contrary. We assume that e contains a vertex $w \notin U$ (see figure 2b).

As in the previous item, we construct the shortest alternating chain P from S to u (in the case $u \in S$ this chain consists of one vertex). Let P goes through the edge uv. Since we have chosen the shortest chain to u, we have $v = a_i$, $u = b_i$ for some $i < n$. Then we have $v \in U$ and we arrive at a contradiction.

Thus the chain P does not go through the edge uv. We add v and w to P and obtain $w \in U$, that contradicts our assumption. Hence v is the only vertex of the hyperedge e that does not belong to U (see figure 2c).
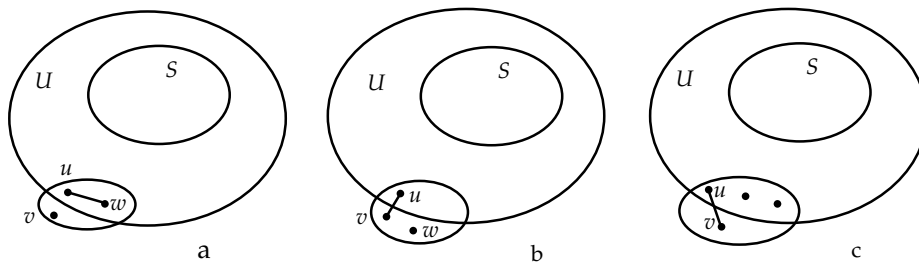

Figure 2: Hyperedges, intersecting U.

3. *For any vertex $u \in U$ we have $d_G(u) \geq k$.*

Let $u \in U$ and $d_G(u) \leq k - 1$. Clearly, $u \notin S$. Consider an alternating chain $P = a_0 b_0 \ldots a_n$ from S to $u = a_n$. We construct a new graph G': take the graph G and for every $i \in [0, n-1]$ replace in the hyperedge $e_i \supset \{a_i, b_i, a_{i+1}\}$ the edge $a_i b_i$ by the edge $b_i a_{i+1}$. It is easy to see that resulting graph G' is also an image of the hypergraph $\mathcal{H}$.

Since $d_{G'}(u) = d_G(u) + 1 \leq k$, then $u \notin V_{k+1}(G')$. For any other vertex x we have $d_{G'}(x) \leq d_G(x)$. Hence $V_{k+1}(G') \subseteq S = V_{k+1}(G)$. It remains to notice that $a_0 \in S$ and $d_G(a_0) > d_{G'}(a_0)$, consequently, $s_{k+1}(G') < s_{k+1}(G)$. We obtain a contradiction with the minimality of $s_{k+1}(G)$.

4. *We estimate the sum of degrees in the hypergraph $\mathcal{H}$ taken over vertices in U.*

Let $u_1, \ldots u_\ell$ be all vertices of U that have degrees less than k in the induced

subgraph $F = G(U)$. We set

$$t_i = d_G(u_i) - d_F(u_i), \quad t = \sum_{i=1}^{\ell} t_i.$$

The degree of any vertex of U in G is at least k due to item 3. Since $S \subset U$, the set U contains vertices that have degrees more than k in G. Hence

$$e(F) = \frac{1}{2} \sum_{u \in U} d_F(u) > \frac{k|U| - t}{2}.$$

We further estimate $m = \sum_{u \in U} d_{\mathcal{H}}(u)$. By item 1, all hyperedges of $\mathcal{H}$ that correspond to the edges of F (i.e. of the set $\varphi(E(F))$), are contained in the set U and contribute to m at least

$$\delta \cdot e(F) > \delta \cdot \frac{k|U| - t}{2} \geq \Delta|U| - \frac{\delta t}{2}.$$

Now we consider t edges of G between U and $V(\mathcal{H}) \setminus U$. According to item 2 each of these edges is contained in a hyperedge of $\mathcal{H}$, which has only one vertex outside U, and, consequently, which has at least $\delta - 1$ vertices in U. We note that all these t hyperedges are different. Thus

$$m > \Delta|U| - \frac{\delta t}{2} + (\delta - 1)t > \Delta|U|.$$

Hence there is a vertex $u \in U$ of degree $d_{\mathcal{H}}(u) > \Delta$, that contradicts to the conditions of the lemma.

The obtained contradiction shows that there exists an image G of $\mathcal{H}$ with $\Delta(G) \leq k$. $\qquad\square$

3 Proofs of Theorems 1 and 2

Proof of theorem 1. 1) We pick an image G of $\mathcal{H}$ with $\Delta(G) \leq k = \lceil \frac{2\Delta}{\delta} \rceil$, which exists due to the lemma 1. Clearly, there exists a proper vertex coloring of the graph G in $k + 1$ colors.

We need to show that this coloring is a proper vertex coloring of the hypergraph $\mathcal{H}$. Let φ be the bijection of the image G. For every hyperedge $e \in E(\mathcal{H})$ we have $\varphi^{-1}(e) \subset e$, and, therefore, two vertices of $\varphi^{-1}(e) \subset E(G)$ have different colors.

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2) To prove the second statement it suffices to find an image of the hypergraph $\mathcal{H}$ that has a proper vertex coloring in k colors for $k \geq 3$ and $\delta \geq 3$. At first we consider an image G of $\mathcal{H}$ with $\Delta(G) \leq k$ and its bijection φ.

We remind the classic Brooks theorem: *if $\Delta(G) \leq k$, $k \geq 3$, and no connected component of G is a clique on $k + 1$ vertices, then G has a proper vertex coloring in k colors.*

Let G have connected components that are cliques on $k + 1$ vertices. We enumerate them all by $C_1,\ldots,$ C_q (for conciseness, we will refer to these components simply by *cliques*). Graph G can possibly have other connected components. We denote by $D_{q+1},\ldots,D_p$ induced subgraphs on these components. In what follows we correct the graph-image G, such that obtained graph would have proper vertex coloring in k colors.

Image transformation.
Consider an arbitrary edge u_iw_i in each clique C_i. It is clear that there is a vertex $v_i \in e_i = \varphi(u_iw_i)$ different from u_i and w_i. We construct the new image G' of $\mathcal{H}$, by replacing simultaneously every edge u_iw_i by the edge u_iv_i. We call the edges $u_1v_1,$ $\ldots,$ u_qv_q by *new edges*.

Further we prove that G' has a proper vertex coloring in k colors.
We construct an auxiliary digraph F: vertices of F are connected components of G, from each component-clique C_i an oriented edge (*arc*) leads to a component that contains v_i. If v_i is a vertex of the clique C_i, then this arc will be a loop. In fact, in order to construct F from G', one could orient the new edges and contract each connected component of G into a vertex.

Our algorithm for coloring vertices in k colors works according to the following plan:

— if there exists a clique that has no incoming arc in F, we perform Step 1 and return to the beginning of the algorithm;

— if each clique has at least one incoming arc in F, then we perform Step 2 and terminate the algorithm.

1. *There is clique C_i that has no incoming arc.*
In this case $d_{G'}(w_i) = k - 1$. We enumerate vertices of C_i starting from w_i and finishing at the vertex u_i that is adjacent in G' to a vertex of another connected component of G. We assume that vertices of the rest components are properly colored in k colors. Then we can color vertices of C_i in the reverse order (respect to their numbers): at each step we take a vertex that is adjacent to less than k already colored vertices and we color it in any remaining color.

Therefore, we can delete from G' all vertices of the component C_i and continue by coloring the remaining graph $G' - C_i$. In addition, we change the graph F. We delete from F vertex C_i and the arc going from C_i.

2. *Every component-clique has an incoming arc.*

Since exactly one arc goes from each clique, then exactly one arc comes into each clique. Thus all cliques in F are divided into several oriented cycles, which vertices are not adjacent to each other in G'. We color these cycles independently. The rest connected components of G (not cliques on $k + 1$ vertices) are the same connected components in G'. Due to Brooks theorem their vertices can be properly colored in k colors.

Now we have cliques $C_1, \ldots, C_\ell$ forming in F an oriented cycle. We denote by G^* the induced subgraph of G' on the union of all these cliques. It remains to prove that G^* has a proper vertex coloring in k colors. If $\Delta(G^*) \leq k$, then it follows from Brooks theorem, since the graph G^* is connected and is not a clique on $k + 1$ vertices. Assume, that $\Delta(G^*) > k$ and consider two cases.

2.1. $\ell = 1$, *i.e. our cycle is a loop and $v_1 \in V(C_1)$.*

Then G^* is a clique on $k + 1$ vertices with deleted edge $u_1 w_1$ and edge $u_1 v_1$ of multiplicity two. Clearly, G^* has a proper coloring in k colors: we color u_1 and w_1 in the same color, and we color each other vertex in its own color.

2.2. $\ell \geq 2$.

Let G^* have a vertex x of degree more than k and x belongs to the clique C_i. Clearly, x is adjacent to a vertex of the clique C_{i-1} and $x \neq w_i$. Moreover, in this case $d_G(w_i) = k - 1$. We can delete the vertex w_i from G^*, since we can color this vertex after coloring the rest vertices. If $x \neq u_i$, then x is adjacent to w_i, hence, all remaining vertices of C_i have degrees not more than k in $G^* - w_i$ (see figure 3a).
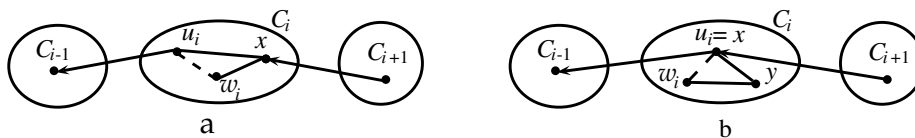

Figure 3: Coloring of the clique C_i in the graph G^*.

If $x = u_i$, then there is another vertex y in $C_i - w$ and $d_{G^* - w_i}(y) = k - 1$. We delete y from the graph (after coloring all other vertices in k colors we can easily color y). Clearly, in the graph $G^* - w_i - y$ the degrees of all remaining vertices of C_i do not exceed k (see figure 3b).

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We perform such operations with all components-cliques that have in G^* a vertex of degree more than k. As a result we obtain a connected graph H^* with maximal degree not exceeding k.

We prove that H^* is not a clique on $k + 1$ vertices.
It is clear from the construction that all new edges between the components $C_1, \ldots, C_n$ remain in H^* after deletion described above (we have not deleted none of their ends from G^*). We consider a component C_2 and two new edges u_1v_1 and u_2v_2, incident to vertices of C_2. Clearly, the graph $H^* - u_1v_1 - u_2v_2$ is disconnected (C_2 is separated from the rest vertices of the graph). Thus H^* becomes disconnected after deleting two of its edges, and, hence, it can't be a clique on $k + 1 \geq 4$ vertices.

Applying Brooks theorem we get a proper k-coloring of H^*. After that we can add back all deleted from G^* vertices and color them properly in reverse order to their deletion.

Thus the graph G' admits a proper k-coloring and this coloring, as it was mentioned above, provides a proper vertex coloring of the hypergraph $\mathcal{H}$. $\square$

Proof of theorem 2. We construct the following hypergraph $\mathcal{H}$. Its vertex set $V(\mathcal{H})$ coincides with $V(G)$; set of hyperedges $E(\mathcal{H})$ consists of neighborhoods $N_G(v)$ of all vertices $v \in V(G)$. Each hyperedge of $\mathcal{H}$ has the size at least $\delta(G)$ and each vertex of $\mathcal{H}$ belongs to not more than $\Delta(G)$ hyperedges. Now it is easy to see, that the statement we are proving is an immediate consequence of theorem 1 applied to the hypergraph $\mathcal{H}$. $\square$

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