Computing Approximate Pure Nash Equilibria in Congestion Games

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Among other solution concepts, the notion of the pure Nash equilibrium plays a central role in Game Theory. Pure Nash equilibria in a game characterize situations with non-cooperative deterministic players in which no player has any incentive to unilaterally deviate from the current situation in order to achieve a higher payoff. Unfortunately, it is well known that there are games that do not have pure Nash equilibria. Furhermore, even in games where the existence of equilibria is guaranteed, their computation can be a computationally hard task. Such negative results significantly question the importance of pure Nash equilibria as solution concepts that characterize the behavior of rational players. Approximate pure Nash equilibria, which characterize situations where no player can significantly improve her payoff by unilaterally deviating from her current strategy, could serve as alternative solution concepts provided that they exist and can be computed efficiently. In this letter, we discuss recent positive algorithmic results for approximate pure Nash equilibria in congestion games.

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1. PROBLEM STATEMENT AND RELATED WORK

In a weighted congestion game, players compete over a set of resources. Each player has a positive weight. Each resource incurs a latency to all players that use it; this latency depends on the total weight of the players that use the resource according to a resource-specific, non-negative, and non-decreasing latency function. Among a given set of strategies (over sets of resources), each player aims to select one selfishly, trying to minimize her individual total cost, i.e., the sum of the latencies on the resources in her strategy. Typical examples include congestion games in

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networks, where the network links correspond to the resources and each player has alternative paths that connect two nodes as strategies.

The case of unweighted congestion games (i.e., when all players have unit weight) has been widely studied in the literature. Rosenthal [1973] proved that these games admit a potential function with the following remarkable property: the difference in the potential value between two states (i.e., two snapshots of strategies) that differ in the strategy of a single player is equal to the difference of the cost experienced by this player in these two states. This immediately implies the existence of a pure Nash equilibrium. Any sequence of improvement moves by the players strictly decreases the value of the potential and a state corresponding to a local minimum of the potential will eventually be reached; this corresponds to a pure Nash equilibrium. For weighted congestion games, potential functions are known only in special cases (e.g., when the latency functions are linear; see [Fotakis et al. 2005]). Actually, in games with polynomial latency functions (e.g., quadratic), pure Nash equilibria may not even exist [Goemans et al. 2005].

Potential functions provide only inefficient proofs of existence of pure Nash equilibria. Fabrikant et al. [2004] proved that the problem of computing a pure Nash equilibrium in a (unweighted) congestion game is PLS-complete (informally, as hard as it could be, given that there is an associated potential function). One consequence of PLS-completeness results is that almost all states in some congestion games are such that any sequence of players' improvement moves that originates from these states and reaches pure Nash equilibria is exponentially long. Efficient algorithms are known only for special cases. For example, Fabrikant et al. [2004] show that the Rosenthal's potential function can be (globally) minimized efficiently by a flow computation in unweighted congestion games in networks when the strategy sets of the players are symmetric.

The above negative results have led to the study of the complexity of approximate pure Nash equilibria. A ρ-approximate (pure Nash) equilibrium is a state, from which no player has an incentive to deviate so that she decreases her cost by a factor larger than ρ. The only positive result that appeared before our recent work is due to Chien and Sinclair [2011] and applies to symmetric unweighted congestion games: under mild assumptions on the latency functions and on the participation of the players in the dynamics, the $(1+\epsilon)$-improvement dynamics converges to a $(1+\epsilon)$-approximate equilibrium after a polynomial number of steps. For non-symmetric (unweighted) congestion games with more general latency functions, Skopalik and Vöcking [2008] show that the problem is still PLS-complete for any polynomially computable ρ.

2. OUR CONTRIBUTION

In two recent papers [Caragiannis et al. 2011; 2012], we present algorithms for computing $O(1)$-approximate equilibria in unweighted and weighted non-symmetric congestion games with polynomial latency functions of constant maximum degree. Our algorithm for unweighted congestion games is presented in [Caragiannis et al. 2011]. It computes $(2+\epsilon)$-approximate pure Nash equilibria in games with linear latency functions, and $d^{d+o(d)}$ approximate equilibria for polynomial latency functions of maximum degree d. The algorithm is surprisingly simple. Essentially, starting from an initial state, it computes a sequence of best-response player moves of length that is bounded by a polynomial in the number of players and $1/\epsilon$. The sequence consists of phases so that the players that participate in each phase experience costs that are polynomially related. This is crucial in order to obtain convergence in polynomial time. Another interesting part of our algorithm is that, within each phase, it coordinates the best response moves according to two different (but simple) criteria; this is the main tool that guarantees that the effect of a phase to previous ones is negligible and, eventually, an approximate equilibrium is reached.

In [Caragiannis et al. 2012], we significantly extend our techniques and obtain an algorithm that computes $O(1)$-approximate equilibria in weighted congestion games. For games with linear latency functions, the approximation guarantee is $\frac{3+\sqrt{5}}{2}+\epsilon$ for arbitrarily small $\epsilon > 0$; for latency functions of maximum degree $d \geq 2$, it is $d^{2d+o(d)}$. These results are much more surprising than they look at first glance. Given that weighted congestion games with superlinear latency functions do not admit potential functions, it is not even clear that $O(1)$-approximate equilibria exist. In order to bypass this obstacle, we introduce a new class of potential games (that we call Ψ-games), which "approximate" weighted congestion games with polynomial latency functions in the following sense. Ψ-games of degree 1 are linear weighted congestion games. Each weighted congestion game of degree $d \geq 2$ has a corresponding Ψ-game of degree d defined in such a way that any ρ-approximate equilibrium in the latter is a $d!\rho$-approximate equilibrium for the former. As an intermediate new result, we obtain that weighted congestion games with polynomial latency functions of degree d have $d!$-approximate equilibria. Our algorithm is actually applied to Ψ-games; it has a simple general structure similar to our algorithm for unweighted games but has also important differences that are due to the dependency of the cost of each player on the weights of other players. Again, the algorithm essentially identifies a best-response sequence of player moves in the Ψ-game that leads to an approximate equilibrium; its length is now polynomial in terms of the number of bits in the representation of the game and $1/\epsilon$.

In both cases, the approximation guarantee is marginally higher than a quantity that characterizes the potential function of the game; this quantity (which we call the stretch) is defined as the worst-case ratio of the potential value at an almost exact pure Nash equilibrium over the globally optimum potential value. For example, the stretch is almost 2 for linear unweighted congestion games, $\frac{3+\sqrt{5}}{2}$ for linear weighted congestion games, and $d^{d+o(d)}$ for Ψ-games of degree $d \geq 2$. A more thorough literature review, the detailed description of the algorithms, and the analysis details can be found in [Caragiannis et al. 2011; 2012].

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