Refining the Cost of Cheap Labor in Set System Auctions

Ning Chen, Edith Elkind, and Nick Gravin

Division of Mathematical Sciences, Nanyang Technological University, Singapore

Abstract. In *set system auctions*, a single buyer needs to purchase services from multiple competing providers, and the set of providers has a combinatorial structure; a popular example is provided by shortest path auctions [1, 7]. In [3] it has been observed that if such an auction is conducted using first-price rules, then, counterintuitively, the buyer's payment may go down if some of the sellers are prohibited from participating in the auction. This reduction in payments has been termed "the cost of cheap labor". In this paper, we demonstrate that the buyer can attain further savings by setting lower bounds on sellers' bids. Our model is a refinement of the original model of [3]: indeed, the latter can be obtained from the former by requiring these lower bounds to take values in $\{0, +\infty\}$. We provide upper and lower bounds on the reduction in the buyer's payments in our model for various set systems, such as minimum spanning tree auctions, bipartite matching auctions, single path and k-path auctions, vertex cover auctions, and dominating set auctions. In particular, we illustrate the power of the new model by showing that for vertex cover auctions, in our model the buyer's savings can be linear, whereas in the original model of [3] no savings can be achieved.

1 Introduction

Combinatorial procurement auctions, or *set system auctions*, play an important role in electronic commerce [13]. In such auctions, a buyer (center) needs to purchase products or services from a number of competing sellers, and the subsets of sellers that satisfy the buyer's requirements can be characterized combinatorially. A well-known example is provided by path auctions [1, 7], where the buyer's aim is to obtain a path in a network whose edges are owned by selfish agents; other examples include minimum spanning tree auctions [14], bipartite matching auctions [3], and vertex cover auctions [2]. An important research goal in this setting is the minimization of the buyer's total payment. While most of the work on this topic focuses on dominant-strategy incentive compatible mechanisms (e.g., [1, 14, 9, 7, 5]), the properties of Nash equilibria of first-price auctions have recently received a lot of attention as well [8, 6, 3].

An interesting — and, perhaps, counterintuitive — property of set system auctions is that the buyer can lower her total payment by prohibiting some of the agents from participating in the auction. In other words, reducing competition in the market can benefit the buyer. This has been observed for VCG mechanisms by Elkind [5] in the context of path auctions (see also [4]). Later, Chen and Karlin [3] discovered that this can also happen in first-price auctions for a variety of set systems. They labeled this phenomenon "the cost of cheap labor", and provided tight bounds on the cost of cheap labor in several set systems.

Prohibiting an agent from participating in an auction can be interpreted as requiring him to raise his bid to $+\infty$. The goal of this paper is to explore a more general approach, namely, allowing the center to place arbitrary lower bounds on all sellers' bids, in a manner reminiscent of using reserve prices in combinatorial auctions. Clearly, this technique is more flexible than simply deleting agents, and hence the resulting savings, which we term "the refined cost of cheap labor", may be even higher than the cost of cheap labor, as defined in [3]. In this paper, we study the benefits of this approach by quantifying the refined cost of cheap labor for a number of well-known set systems.

We start by providing general upper and lower bounds on the refined cost of cheap labor for arbitrary set systems (Section 3). We then consider several classes of set systems for which we can show that the refined cost of cheap labor and the cost of cheap labor coincide. These include matroids and (single) paths considered in [3], as well as a richer set system not considered in [3], namely, k-paths. For k-path auctions, we significantly extend the techniques of [3] to provide tight bounds on the (refined) cost of cheap labor.

We then move on to vertex cover set systems. In these set systems, deleting an agent creates a monopoly, and hence the cost of cheap labor is exactly 1. On the other hand, artificially inflating the agents' bids may prove to be very profitable for the buyer: we show that there exist vertex cover auctions for which the refined cost of cheap labor is linear in the number of agents, matching the general upper bound of Section 3. Finally, we consider set systems that are based on dominating sets and perfect bipartite matchings. For such set systems, we show that both the cost of cheap labor and the refined cost of cheap labor can be quite large, and also that these two quantities can differ by a large factor. These set systems illustrate that setting lower bounds on the sellers' bids is a very powerful — yet simple and practically applicable — technique. Thus, we believe that the refined cost of cheap labor is an important characteristic of a set system auction, which deserves further study.

2 Preliminaries

A *set system* is a pair $(E, \mathcal{F})$, where E is the *ground set* and $\mathcal{F} \subseteq 2^E$ is a collection of *feasible* subsets of E. Throughout the paper, we only consider set systems with $|E| < +\infty$ and set $n = |E|$. The set $\mathcal{F}$ can be listed explicitly, or defined combinatorially. In this paper, we consider the following set systems:

- *spanning trees*: the set E is the set of all edges of a given graph G and $\mathcal{F}$ consists of all sets $S \subseteq E$ that contain a spanning tree. This is a special case of a more general *matroid* set system [12], in which the set E is the ground set of a given matroid M, and the set $\mathcal{F}$ is the collection of all subsets of 2^E that contain a base of M.
- *perfect bipartite matchings*: the set E is the set of all edges of a given bipartite graph G and $\mathcal{F}$ consists of all sets $S \subseteq E$ that contain a perfect bipartite matching.
- *k-paths*: the set E is the set of all edges of a given network G with a source s and a sink t, and $\mathcal{F}$ consists of all sets $S \subseteq E$ that contain k edge-disjoint s-t paths.

- *vertex covers*: the set E is the set of all vertices of a given graph G, and $\mathcal{F}$ consists of all sets $S \subseteq E$ that contain a vertex cover of G.
- *dominating sets*: the set E is the set of all vertices of a given graph G, and $\mathcal{F}$ consists of all sets $S \subseteq E$ that contain a dominating set of G, i.e., for each vertex $v \notin S$, there is $u \in S$ such that there is an edge between u and v.

Observe that all set systems listed above are *upwards closed*, i.e., $S \in \mathcal{F}$ implies $S' \in \mathcal{F}$ for any $S' \supseteq S$. A set system is said to be *monopoly-free* if $\cap_{S \in \mathcal{F}} S = \emptyset$. Throughout this paper, we restrict ourselves to upwards closed, monopoly-free set systems.

In a *set system auction* for a set system $(E, \mathcal{F})$, each $e \in E$ is owned by a selfish agent, and there exists a center (auctioneer) who wants to purchase a feasible solution, i.e., an element of $\mathcal{F}$. Each agent $e \in E$ has a cost $c_e \geq 0$, which is incurred if this element is used in the solution purchased by the center. We will refer to a triple $(E, \mathcal{F}, \mathbf{c})$, where $\mathbf{c} = (c_e)_{e \in E}$ as a *market*. For any subset $S \subseteq E$, we write $c(S)$ to denote $\sum_{e \in S} c_e$.

Throughout the paper, we assume that the sale is conducted by means of a first-price auction: each agent e announces his *bid* b_e, indicating how much he wants to be paid for the use of his element, the auctioneer selects the cheapest feasible set breaking ties in an arbitrary (but deterministic) way, and all agents in the winning set are paid their bid. Thus, the payoff of a winning agent e with bid b_e is $b_e - c_e$, whereas the payoff of any losing agent is 0. The agents are selfish, i.e., they aim to maximize their payoff. Therefore, we are interested in *Nash equilibria (NE)* of such auctions, i.e., vectors of bids $\mathbf{b} = (b_e)_{e \in E}$ such that no agent e can increase his payoff by bidding $b'_e \neq b_e$ as long as all other agents bid according to $\mathbf{b}$. We restrict ourselves to equilibria in which no agent bids below their cost, i.e., $b_e \geq c_e$ for all $e \in E$.

Unfortunately, as shown in [8], for some markets and some tie-breaking rules, first-price auctions may have no NE in pure strategies. However, they do have *ε-Nash equilibria* in pure strategies for any $\varepsilon > 0$, i.e., a bid vector such that no agent can unilaterally change his bid to increase the payoff by more than ε. Moreover, for any market $(E, \mathcal{F}, \mathbf{c})$, there exists a tie-breaking rule (e.g., one that favors the feasible set with the smallest cost) that ensures the existence of a pure NE. Thus, in what follows, we will ignore the issues of existence of pure NE, and use the term "Nash equilibrium" to refer to a bid vector that is a pure NE of a first-price auction for a given market under *some* tie-breaking rule, or, equivalently, can be obtained as a limit of ε-NE for that market as $\varepsilon \to 0$.

Following the approach of [3], we will focus on NE of set system auctions that are *buyer-optimal*, i.e., minimize the center's total payment. For a given market $(E, \mathcal{F}, \mathbf{c})$, we denote the center's total payment in such a NE with the smallest total payment by $\nu(E, \mathcal{F}, \mathbf{c})$.

This quantity is similar to—but different from—the quantity ν_0 that is used in [9] as a benchmark to measure the frugality of dominant-strategy set system auctions. Indeed, the latter can be interpreted as the minimal total payment in a buyer-optimal *efficient* NE of a first-price auction (i.e., a NE in which the winning set S satisfies $S \in \mathrm{argmin}_{S \in \mathcal{F}} c(S)$), in which, in addition, all losing agents bid their cost.

3 Refined Cost of Cheap Labor: General Bounds

In this section, we introduce our new measure of the cost of cheap labor, which we will call the *refined cost of cheap labor*, and compare it to the notion of the cheap labor cost introduced in [3].

The following definition of cheap labor cost is adapted from [3].

Definition 1. *Given a market* $(E, \mathcal{F}, \mathbf{c})$*, its* cheap labor cost $\delta_1(E, \mathcal{F}, \mathbf{c})$ *is defined as follows:*

$$\delta_1(E, \mathcal{F}, \mathbf{c}) = \max_{S \subseteq E} \frac{\nu(E, \mathcal{F}, \mathbf{c})}{\nu(S, \mathcal{F}[S], \mathbf{c}[S])},$$

where $\mathcal{F}[S] = \{S' \in \mathcal{F} \mid S' \subseteq S\}$*, and* $\mathbf{c}[S] = (c_e)_{e \in S}$*. The* cheap labor cost of a set system $(E, \mathcal{F})$ *is defined as* $\delta_1(E, \mathcal{F}) = \sup_{\mathbf{c}} \delta_1(E, \mathcal{F}, \mathbf{c})$.

Informally, $\delta_1(E, \mathcal{F})$ measures how much the center can save by removing some of the agents from the system. Alternatively, the center's actions can be interpreted as setting the costs of some agents to $+\infty$ (or some appropriately large number). The notion of *refined cheap labor cost*, which we will now introduce, allows the center more flexibility, permitting him to raise the cost of any agent $e \in E$ to *any* value between its cost c_e and $+\infty$.

Definition 2. *Given a market* $(E, \mathcal{F}, \mathbf{c})$*, its* refined cheap labor cost $\delta_2(E, \mathcal{F}, \mathbf{c})$ *is defined as follows:*

$$\delta_2(E, \mathcal{F}, \mathbf{c}) = \sup_{\mathbf{c}' \succeq \mathbf{c}} \frac{\nu(E, \mathcal{F}, \mathbf{c})}{\nu(E, \mathcal{F}, \mathbf{c}')}.$$

where $\mathbf{c}' \succeq \mathbf{c}$ *means that* $c'_e \geq c_e$ *for all* $e \in E$*. The* refined cheap labor cost of a set system $(E, \mathcal{F})$ *is defined as* $\delta_2(E, \mathcal{F}) = \sup_{\mathbf{c}} \delta_2(E, \mathcal{F}, \mathbf{c})$.

As argued above, Definition 1 can be obtained from Definition 2 by requiring that $c'_e \in \{c_e, +\infty\}$ for all $e \in E$. The following theorem provides some simple bounds on δ_1 and δ_2.

Theorem 1. *Fix a market* $(E, \mathcal{F}, \mathbf{c})$*, and let* S *be a cheapest feasible solution in* $\mathcal{F}$ *with respect to* $\mathbf{c}$*. Then the following inequalities hold:*

$$1 \leq \delta_1(E, \mathcal{F}, \mathbf{c}) \leq \delta_2(E, \mathcal{F}, \mathbf{c}) \leq |S|.$$

In what follows, we present upper and lower bounds on δ_2 for specific set systems.

4 Spanning Trees and Other Matroids

For any spanning tree set system, artificially inflating the agents' costs cannot lower the center's payments, i.e., $\delta_1 = \delta_2 = 1$ (where $\delta_1 = 1$ is shown in [3]). In fact, this result holds for the more general case of matroid set systems. We refer the readers to [12] for a formal definition of a matroid.

Theorem 2. *For any matroid market* $\mathcal{M} = (E, \mathcal{F}, \mathbf{c})$ *we have* $\delta_2(E, \mathcal{F}, \mathbf{c}) = 1$.

5 Paths and k-Paths

Throughout this section, for a given network $G = (V, E)$ with a source s and a sink t, we denote by $\mathcal{F}_k$ the collection of sets of edges that contain k edge-disjoint paths from s to t.

For k-paths set systems, it turns out that the optimal cost reduction can be achieved by simply deleting edges in E, i.e., $\delta_1 = \delta_2$. Furthermore, $\delta_2 = \delta_1 \leq k + 1$ for any network, and this bound is tight, i.e. for any k there is a k-path set system $(E, \mathcal{F}_k)$ with $\delta_1(E, \mathcal{F}_k) = \delta_2(E, \mathcal{F}_k) = k + 1$. This generalizes the result of [3], which proves this claim for $k = 1$.

Theorem 3. *For any network $G = (V, E)$ with a source s and a sink t, and any cost vector $\mathbf{c}$, we have $\delta_2(E, \mathcal{F}_k, \mathbf{c}) = \delta_1(E, \mathcal{F}_k, \mathbf{c})$.*

We also give a tight bound on the cost of cheap labor (and hence, by Theorem 3, a tight bound on the refined cost of cheap labor) in any k-paths set system.

Theorem 4. *For any network $G = (V, E)$ with a source s and a sink t, and any cost vector $\mathbf{c}$, we have $\delta_1(E, \mathcal{F}_k, \mathbf{c}) \leq k + 1$, and this bound is tight.*

6 Vertex Covers

In this section, we consider vertex cover auctions. In these auctions, as well as in the auctions considered in Section 7, the sellers are the vertices. Therefore, in these two sections we depart from the standard graph-theoretic notation, and use E to denote the set of vertices of a graph G, and H to denote the set of edges of G. Also, we denote by $\mathcal{F}$ the collection of all sets of vertices that contain a vertex cover (respectively, a dominating set) for G.

The vertex cover set systems demonstrate that δ_1 and δ_2 can be very different: for any such set system $\delta_1 = 1$, whereas δ_2 can be linear in $|E|$.

Proposition 1. *For any graph $G = (E, H)$ and any costs $\mathbf{c}$, we have $\delta_1(E, \mathcal{F}, \mathbf{c}) = 1$.*

In contrast, we will now show that there is a graph $G = (E, H)$ with $|E| = n$ such that the corresponding set system $(E, \mathcal{F})$ satisfies $\delta_2(E, \mathcal{F}) = \Omega(n)$.

Proposition 2. *There exists a graph $G = (E, H)$ and a cost vector $\mathbf{c}$ that satisfy $\delta_2(E, \mathcal{F}, \mathbf{c}) \geq \frac{n-3}{2}$, where $n = |E|$.*

Proof. Consider a graph G obtained from complete graph K_{n-2} by adding two new vertices u and u' and connecting them to two adjacent vertices v and v' of K_{n-2}, respectively (see Fig. 1). In addition, consider a cost vector $\mathbf{c}$ given by $c_v = c_{v'} = 1$, and $c_e = 0$ for $e \neq v, v'$.

For the cost vector $\mathbf{c}$, it can be seen that the buyer-optimal NE $\mathbf{b}$ is $b_u = b_{u'} = 0$, $b_e = 1$ for $e \neq u, u'$. Thus, $\nu(E, \mathcal{F}, \mathbf{c}) = n - 3$. On the other hand, consider a cost vector $\mathbf{c}' \succeq \mathbf{c}$ given by $c'_v = c'_{v'} = c'_u = c'_{u'} = 1$ and $c_e = 0$ for $e \neq v, v', u, u'$. It is easy to see that for this cost vector, the buyer-optimal NE $\mathbf{b}'$ satisfies $\mathbf{b}' = \mathbf{c}'$ and the winning set consists of all vertices of K_{n-2}. Hence, $\nu(E, \mathcal{F}, \mathbf{c}') = 2$, and we have $\delta_2(E, \mathcal{F}, \mathbf{c}) \geq \frac{n-3}{2}$.

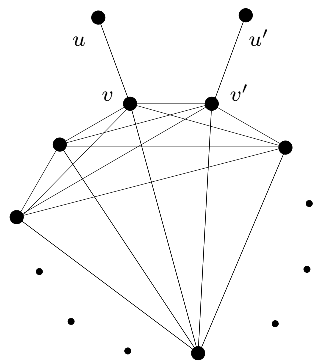

Fig. 1. Vertex Cover

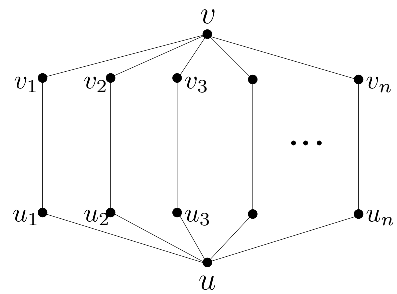

Fig. 2. Perfect Bipartite Matching

7 Dominating Sets

For dominating sets, note that deleting an agent that corresponds to a vertex e is not equivalent to deleting the vertex e itself from the graph: e still needs to be dominated, even though it cannot be a member of a feasible set.

For dominating sets, δ_1 does not necessarily equal δ_2. Furthermore, δ_1 and δ_2 can be as large as $\Omega(\sqrt{n})$. We will now present two examples to illustrate this. Both examples are obtained by a modification of the construction used in the last section.

Definition 3. *Given a complete graph K_n, $n \geq 3$, let K'_n be the graph obtained from K_n by replacing each of its edges (v_i, v_j) by a pair of edges $(v_i, w_{ij}), (w_{ij}, v_j)$. Define $W = \{w_{ij}\}_{i,j \in \{1,\dots,n\}}$ and $V = \{v_i\}_{i \in \{1,\dots,n\}}$.*

Proposition 3. *There exists a graph $G = (E, H)$ and a cost vector $\mathbf{c}$ that satisfy $\delta_1(E, \mathcal{F}, \mathbf{c}) = 1$ and $\delta_2(E, \mathcal{F}, \mathbf{c}) = \Omega(\sqrt{n})$.*

The graph G is constructed from K'_n by selecting two adjacent vertices $v, v' \in V$ and adding three new vertices t, u, u' and $n + 2$ new edges (u, v), (u', v'), $(t, v)_{v \in V}$ (see Fig. 3 [left]). For cost vector $\mathbf{c}$, we set $c_e = n^2$ for $e \in W$, $c_v = c_{v'} = 1$, $c_u = c_{u'} = c_t = 0$, and $c_e = 0$ for $e \in V \setminus \{v, v'\}$.

Proposition 4. *There is a graph $G = (E, H)$ such that $\delta_1(E, \mathcal{F}) = \Omega(\sqrt{n})$ and $\delta_2(E, \mathcal{F}) = \Omega(\sqrt{n})$.*

The graph G is constructed from K'_n by selecting a vertex $v \in V$ and adding three new vertices t, u, u' and $n + 3$ new edges (u, v), (u', v), (u, u'), $(t, v)_{v \in V}$ (see Fig. 3 [right]). For cost vector $\mathbf{c}$, we set $c_e = n^2$ for $e \in W$, $c_v = 1$, $c_u = c_{u'} = c_t = 0$, and $c_e = 0$ for $e \in V \setminus \{v\}$.

8 Perfect Bipartite Matchings

Perfect bipartite matching systems have a similar flavor to dominating set systems—δ_2 can be very different from δ_1, and both of them can be very large. For perfect matching

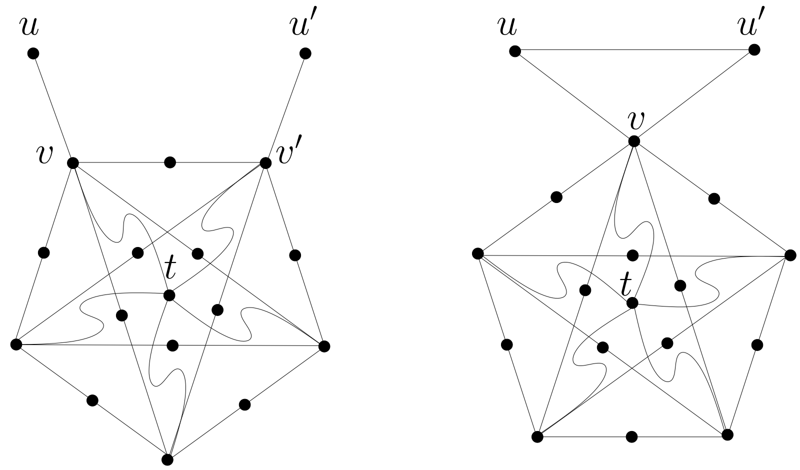

Fig. 3. Dominating Set (graph G [left] and G' [right] with $n = 5$).

in bipartite graphs, [3] shows that there is a graph G such that the corresponding set system satisfies $\delta_1(E, \mathcal{F}) = \Omega(n)$. As any bipartite matching in a graph with n edges has size $O(n)$, by Theorem 1 we have the following claim.

Proposition 5. *There is a graph $G = (V, E)$ such that $\delta_1(E, \mathcal{F}) = \Theta(n)$ and $\delta_2(E, \mathcal{F}) = \Theta(n)$, where $n = |E|$.*

Proposition 5 shows that in the worst case δ_1 and δ_2 coincide. However, they can also differ by a linear factor.

Proposition 6. *There is a graph $G = (V, E)$ such that $\delta_1(E, \mathcal{F}) = 1$ and $\delta_2(E, \mathcal{F}) = \Omega(n)$.*

Proof. Consider the graph shown in Fig. 2. For any cost vector $\mathbf{c}$, since we cannot delete any edge without creating a monopoly, we have $\delta_1(E, \mathcal{F}, \mathbf{c}) = 1$.

On the other hand, to see that $\delta_2(E, \mathcal{F}) = \Omega(n)$, consider a cost vector $\mathbf{c}$ where $c_{(u_i,u)} = 1$ for $i = 3, \ldots, n$, and $c_e = 0$ for any other edge $e \in E$. In any buyer-optimal Nash equilibrium $\mathbf{b}$, we have to set $b_{(u_i,v_i)} = 1$ for $i = 3, \ldots, n$, which implies that $\nu(E, \mathcal{F}, \mathbf{c}) = n - 2$. Consider another cost vector $\mathbf{c}' \succeq \mathbf{c}$, where $c_{(u_i,u)} = 1$ for $i = 1, \ldots, n$ and $c_e = 0$ for any other edge $e \in E$. It can be seen that $\nu(E, \mathcal{F}, \mathbf{c}') = 1$, and thus $\delta_2(E, \mathcal{F}, \mathbf{c}) \geq n - 2$.

9 Conclusions and Future Work

We have introduced the notion of refined cost of cheap labor for set system auctions, and analyzed it for several classes of set systems. A number of questions suggest themselves for further study. First, in this paper we largely ignored computational issues related to our problem, such as, e.g., computing the refined cost of cheap labor for a given set system, or identifying an optimal or close-to-optimal modified cost vector $\mathbf{c}'$. We believe that this is a fruitful topic that deserves to be investigated further. Another promising research direction is bounding the ratio between ν and ν_0, i.e., the additional cost of requiring the winning set to be optimal with respect to the true costs; this quantity can be seen as "the cost of efficiency". In particular, it would be interesting to see if the latter can be bounded in terms of the (refined) cost of cheap labor.

10 Acknowledgements

This work was supported by a NAP grant, NRF grant NRF-RF2009-08, and A-STAR SINGA Graduate Scholarship (Singapore).

References

1. A. Archer, É. Tardos, *Frugal Path Mechanisms*, SODA 2002, 991-999.
2. G. Calinescu, *Bounding the Payment of Approximate Truthful Mechanisms*, ISAAC 2004, 221-233.
3. N. Chen, A. R. Karlin, *Cheap Labor Can Be Expensive*, SODA 2007, 707-715.
4. V. Conitzer, T. Sandholm, *Failures of the VCG Mechanism in Combinatorial Auctions and Exchanges*, AAMAS 2006, 521-528.
5. E. Elkind, *True Costs of Cheap Labor are Hard to Measure: Edge Deletion and VCG Payments in Graphs*, EC 2005, 108-116.
6. E. Elkind, L. A. Goldberg, P. W. Goldberg, *Frugality Ratios and Improved Truthful Mechanisms for Vertex Cover*, EC 2007, 336-345.
7. E. Elkind, A. Sahai, K. Steiglitz, *Frugality in Path Auctions*, SODA 2004, 701-709.
8. N. Immorlica, D. Karger, E. Nikolova, R. Sami, *First-Price Path Auctions*, EC 2005, 203-212.
9. A. R. Karlin, D. Kempe, T. Tamir, *Beyond VCG: Frugality of Truthful Mechanisms*, FOCS 2005, 615-626.
10. V. Krishna, *Auction Theory*, Academic Press, 2002.
11. N. Nisan, A. Ronen, *Algorithmic Mechanism Design*, STOC 1999, 129-140.
12. J. Oxley, *Matroid Theory*, Oxford University Press, 1992.
13. T. Sandholm, *Expressive Commerce and Its Application to Sourcing: How We Conducted $35 Billion of Generalized Combinatorial Auctions*, ICEC 2007, 349-350.
14. K. Talwar, *The Price of Truth: Frugality in Truthful Mechanisms*, STACS 2003, 608-619.